Redshift-Space Distortions: when a bug becomes a feature

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Abstract

Didactical notes about Redshift-Space Distortions.

1 Introduction

In the past 30 years, galaxy surveys have become a prominent cosmological observable, able to probe the large-scale structure formation. Galaxies trace the underlying matter distribution, whose statistics can be predicted by solving the Einstein-Boltzmann system of linear perturbations, and then correcting this result by adding nonlinear effects. When galaxy surveys are made, we measure their angular positions and their redshifts, which are then mapped to radial coordinates using the Friedmann equation and the FRW metric.

One important detail in this process is that we assume that all of the measured redshift comes from the cosmological redshift. In practice, galaxies have peculiar velocities sourced by the underlying matter field. When we directly transform redshifts into distances, the peculiar velocities distort the radial coordinate of galaxies, effectively distorting the galaxy distribution: these are the **Redshift Space Distortions (RSDs)**. This effect is understood in two different regimes, schematically shown in Figure 1:

- Large-scale regime, "pancakes of God": the peculiar velocity field points to the center of large, overdense regions. This makes this overdense region become more squashed along the line-of-sight;
- Small-scale regime, "fingers of God": for collapsed structures, the peculiar velocities are much larger and the size of the collapsed structure is small. The distortion is so large that it "flips" the overdensity inside-out, effectively stretching the overdensity along the line-of-sight.

Figure 2 shows the actual galaxy sample obtained from eBOSS. One can see in the left panel that the galaxy distribution seems to have radial features pointing to the observer. After taking this effect into account, one can reconstruct the real space galaxy density, which looks (and is) statistical homogeneous and isotropic.

Since RSDs are sourced by the peculiar velocity field, they actually carry useful cosmological information. In these notes, I will discuss RSDs more in-depth.

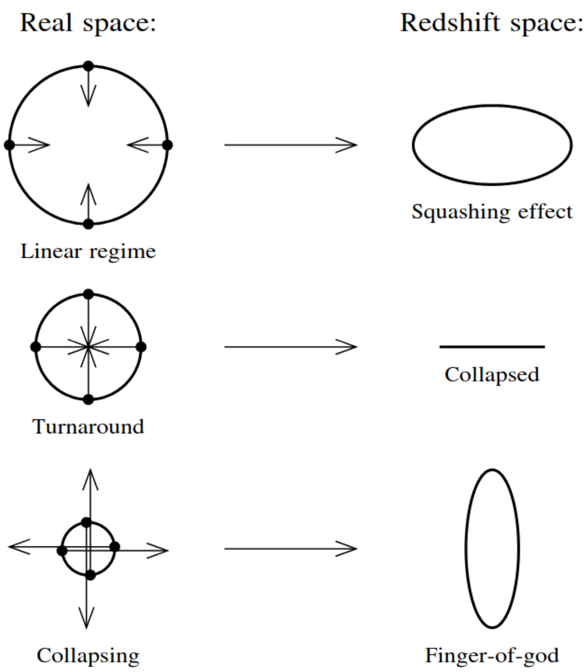

Figure 1: Schematic representation of the redshift-space distortion effect at different scales: the top row shows the large-scale regime, and the bottom row shows the small-scale regime. The left column shows the overdensity in real space, whereas the right column shows the overdensity in redshift space (observed).

2 Redshift-Space Distortions

2.1 The coordinate transformation

The main idea about redshift-space distortions is the use of two coordinate systems: the ”real” coordinate system, whose position will be denoted by $\mathbf{r}$, and the redshift-space coordinated, denoted by $\mathbf{s}$. The relation between the two coordinates is given by

$$\mathbf{s} = \mathbf{r} + v\hat{\mathbf{r}}. \tag{1}$$

Equation 1 is the main equation describing RSDs. A detail about these equations is that, for convenience, distances r will be scaled by the inverse Hubble factor $1/H$, and thus have units of velocity.

A simple derivation goes as follows. Suppose the redshift of a source has a cosmological component z and a peculiar velocity component $z_v = v \ll z$, and hence the total redshift is $z + v$. If we convert that into a comoving radial distance χ^s, where the superscript s denotes redshift space, we obtain

$$\chi^s = \int_0^{z+v} \frac{dz'}{H(z')} \approx \int_0^z \frac{dz'}{H(z')} + \frac{v}{H(z)} = \chi + \frac{v}{H}. \tag{2}$$

Rescaling the distances by the Hubble factor H, we obtain

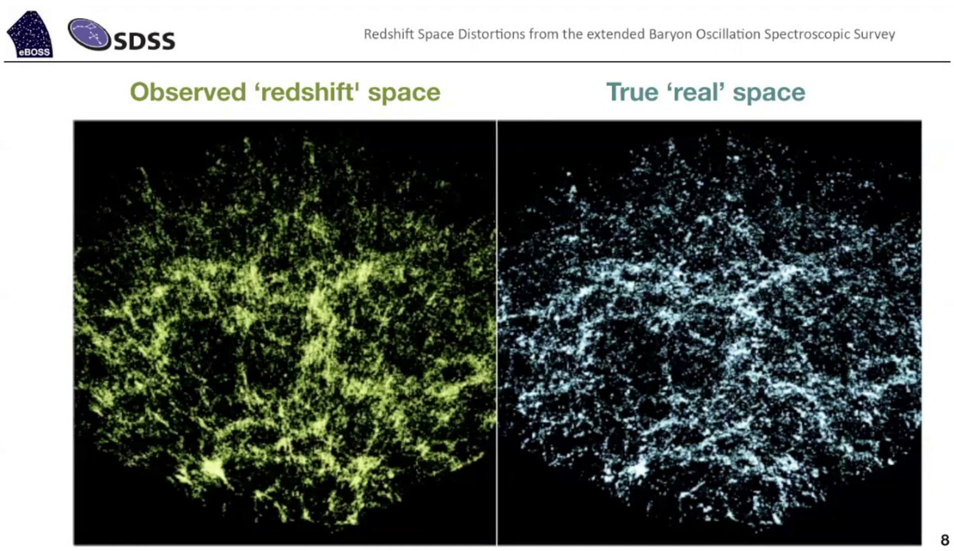

Figure 2: RSD effect from eBOSS galaxies. In the left panel, the redshift-space distribution seems to have radial features that point to the observer, breaking statistical homogeneity. In the right panel, the reconstructed galaxy distribution is free from such features. Credits: Héctor Gil-Marín.

$$\chi^s = \chi + v. \tag{3}$$

Since the Doppler effect does not affect the angular position of the source, it follows that

$$\mathbf{s} = \mathbf{r} + v\hat{\mathbf{r}}. \tag{4}$$

2.2 The density field

The galaxy overdensity field is constructed from the survey via

$$\delta_g = \frac{n(\mathbf{r}) - \bar{n}(\mathbf{r})}{\bar{n}(\mathbf{r})}, \tag{5}$$

where $n(\mathbf{r})$ is the number density of galaxies and $\bar{n}(\mathbf{r})$ is the survey selection function, or the expected number density of galaxies at a given point $\mathbf{r}$. The selection function can be understood as the probability of a galaxy being added to the survey.

The galaxy density field traces the total matter density field. We can model this relation by a simple, constant, linear bias b such that

$$\delta_g = b\delta_m. \tag{6}$$

While there are more complex models for galaxy bias, we can always go back and change this prescription if it's warranted. It's also important to notice that the velocity field is shared between matter and galaxies,

$$\mathbf{v}_g = \mathbf{v}_m. \tag{7}$$

The continuity equation for the matter field is

$$\delta_m' + \nabla \cdot \mathbf{v}_m = 0. \tag{8}$$

The matter perturbations are proportional to the growth function $D(a)$. Therefore, we can write the derivative as

$$\delta_m' = \delta_{m,i} D' = \delta_{m,i} D \frac{d \ln D}{d\tau} = \delta \mathcal{H} f, \tag{9}$$

where f is the growth factor,

$$f = \frac{d \ln D}{d \ln a} \approx [\Omega_m(a)]^\gamma, \tag{10}$$

$\Omega_m(a)$ is the fraction of matter at redshift a, and $\gamma \approx 0.55$. Therefore,

$$\mathcal{H} f \delta_m + \nabla \cdot \mathbf{v}_m. \tag{11}$$

Using the bias relation, we obtain

$$\mathcal{H} \frac{f}{b} \delta_g + \nabla \cdot \mathbf{v}_g = 0. \tag{12}$$

Once again, we can do the trick of absorbing the Hubble factor into the coordinate definition (which is in the $\nabla = \partial/\partial x^i$. Furthermore, we define $\beta = f/b$, obtaining

$$\beta \delta_g + \nabla \cdot \mathbf{v}_g = 0. \tag{13}$$

Taking the gradient of this equation, we get

$$\beta \nabla \delta_g + \nabla^2 \mathbf{v}_g = 0. \tag{14}$$

And then we can "schematically write" the velocity field in terms of the density field as

$$\mathbf{v}_g = -\beta \nabla (\nabla^{-2} \delta_g), \tag{15}$$

where ∇^{-2} is the "inverse Laplacian", or the Green function for the Laplacian. In Fourier space, this relation can be more appropriately written, but we will postpone this process.

Now, we must rewrite the density field δ_g in redshift space using the coordinate transformation, Equation 1. Let's define this new quantity as δ_g^s, where I'm again using the superscript s to denote redshift-space. Notice that we cannot simply write

$$\delta_g^s(\mathbf{s}) = \delta_g(\mathbf{r}) \tag{16}$$

because this way we would not be conserving the number of galaxies, and RSDs are only distortions, they do not alter the number of galaxies. The appropriate way of converting between $\delta_g(\mathbf{r})$ and $\delta_g^s(\mathbf{s})$ is exactly via the galaxy conservation,

$$n_g^s(\mathbf{s}) d^3 s = n_g(\mathbf{r}) d^3 r. \tag{17}$$

Now, it's some lengthy math to obtain the result:

$$
\begin{aligned}
\bar{n}_g^s(\mathbf{s})(1+\delta_g^s(\mathbf{s}))d^3s &= \bar{n}_g(\mathbf{r})(1+\delta_g(\mathbf{r}))d^3r,\\
\bar{n}_g^s(\mathbf{s})(1+\delta_g^s(\mathbf{s}))s^2ds &= \bar{n}_g(\mathbf{r})(1+\delta_g(\mathbf{r}))r^2dr,\\
1+\delta_g^s(\mathbf{s}) &= \frac{\bar{n}_g(\mathbf{r})}{\bar{n}_g^s(\mathbf{s})}\frac{r^2}{s^2}\frac{dr}{ds}(1+\delta_g(\mathbf{r})),\\
\mathbf{s} = \mathbf{r}+v\hat{\mathbf{r}},\ s &= r+v,\\
1+\delta_g^s(\mathbf{s}) &= \frac{\bar{n}_g(\mathbf{r})}{\bar{n}_g(\mathbf{r}+v\hat{\mathbf{r}})}\frac{r^2}{r^2+v^2}\frac{1}{1+\frac{\partial v}{\partial r}}(1+\delta_g(\mathbf{r})),\\
1+\delta_g^s(\mathbf{s}) &= \frac{\bar{n}_g(\mathbf{r})}{\bar{n}_g(\mathbf{r})+\frac{\partial \bar{n}}{\partial r}v}\left(1-2\frac{v}{r}\right)\left(1-\frac{\partial v}{\partial r}\right)(1+\delta_g(\mathbf{r})),\\
1+\delta_g^s(\mathbf{s}) &= \left(1-\frac{\partial \ln \bar{n}}{\partial r}v\right)\left(1-2\frac{v}{r}\right)\left(1-\frac{\partial v}{\partial r}\right)(1+\delta_g(\mathbf{r})),\\
1+\delta_g^s(\mathbf{s}) &= 1+\delta_g(\mathbf{r})-\left(\frac{\partial \ln \bar{n}}{\partial r}+\frac{2}{r}+\frac{\partial}{\partial r}\right)v,\\
\delta_g^s(\mathbf{s}) &= \delta_g(\mathbf{r})-\left(\frac{\partial \ln r^2\bar{n}}{r\partial \ln r}+\frac{\partial}{\partial r}\right)v.
\end{aligned}
$$

Assuming the plane-parallel approximation, where z is the line-of-sight direction, we have

$$\delta^s(\mathbf{s}) = \delta_g(\mathbf{r}) - \frac{\partial}{\partial z}v. \tag{18}$$

Finally, writing the velocity field in terms of the density field, we get

$$\delta_g^s(\mathbf{s}) = \delta_g(\mathbf{r}) + \beta\frac{\partial^2}{\partial z^2}\nabla^{-2}\delta_g. \tag{19}$$

In Fourier space, this can be simply written as

$$\delta_g^s(\mathbf{k}) = \left(1+\beta\frac{k_z^2}{k^2}\right)\delta_g(\mathbf{k}). \tag{20}$$

The density field Fourier modes are increased by $k_z^2/k^2 = \mu^2 = \cos^2\theta$, where k_z is the component of the wavevector parallel to the line-of-sight, and θ is the angle between the wavevector and the line-of-sight. This way, the redshift-space (observed) power spectrum is

$$P^s(\mathbf{k}) = P^s(k, k_z) = (1+\beta\mu^2)^2P(k). \tag{21}$$

Interesting! The power spectrum is not isotropic, but depends on the angle between the wavevector and the line-of-sight. These anisotropies depend on the growth factor f, and therefore they can be used to probe the growth factor (also depending on the galaxy bias b)!

Conversely, the 2-point correlation function is also not isotropic, but depend on the component of the separation vector along the line-of-sight. This was first observed in the 2-degree Field Galaxy Survey (2dFGS) in 2001, and shown in Figure 3.

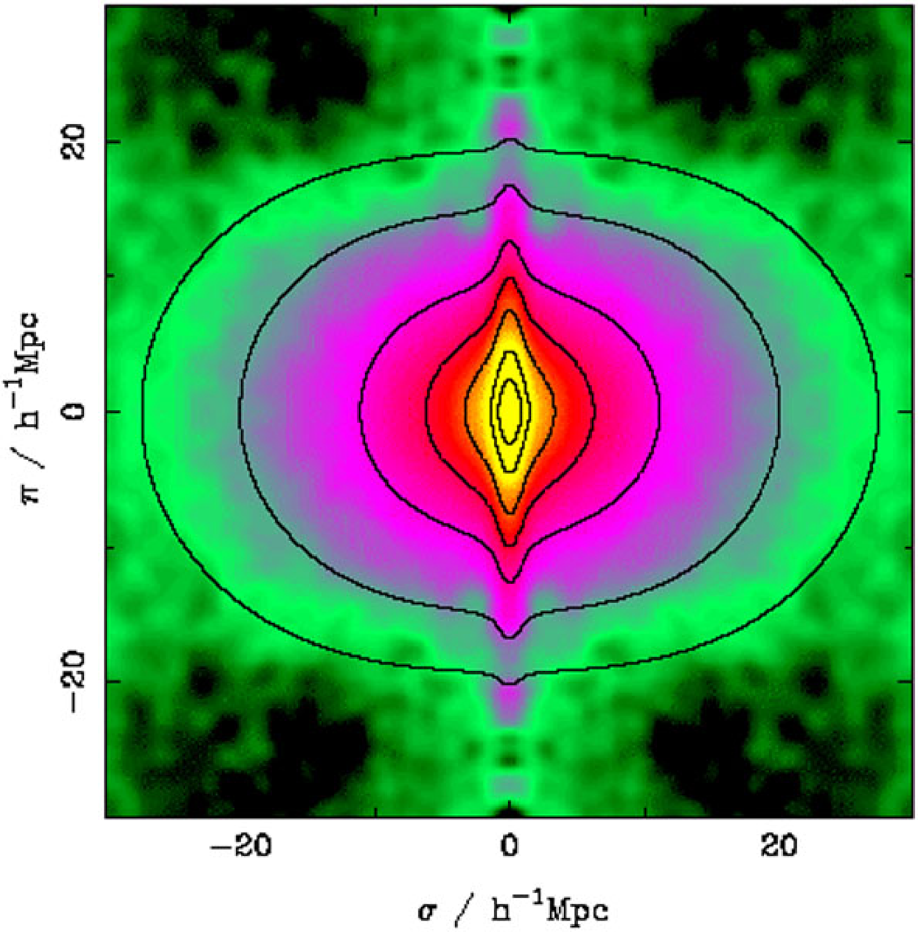

Figure 3: 2-point correlation function of galaxies, measured by the 2-degree Field Galaxy Survey. The horizontal axis is the separation perpendicular to the line-of-sight, while the vertical axis is the separation along the line-of-sight. Colors denote the magnitude of the correlation function, and black lines are the predictions for $\beta \approx 0.45$. One can see the two regimes simultaneously: at large separations, the correlation function is squashed along the LOS, while at small separations, the correlation is stretched along the LOS.

3 Takeaways

- Peculiar velocities of astronomical objects affect their redshift: not taking this into account distorts the observed distribution;
- RSDs squash large-scale distributions along the line-of-sight, while elongating small-scale distributions along the line-of-sight;
- RSDs break the statistical isotropy in a predictable way, Equation 21;
- RSDs can probe the growth factor f.

4 References

- Original work about RSD is from Nick Kaiser (1987), `https://articles.adsabs.harvard.edu/pdf/1987MNRAS.227....1K`,
- Review about RSD from A. Hamilton (1997), `https://arxiv.org/pdf/astro-ph/9708102`
- The first measurement of RSD from 2dF galaxies, `https://arxiv.org/pdf/astro-ph/0103143`